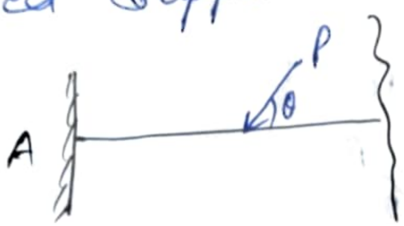
# Structural Analysis

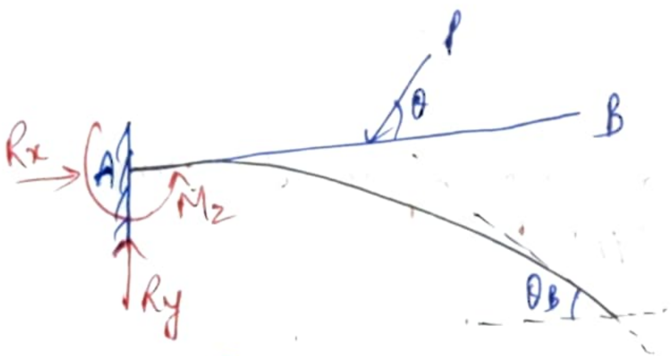
## INDETERMINACY

Restraining the displacement will lead to the formation of external support Reactions.

### I. External Support Reactions

① Fixed Support




$\Delta x = 0$

$\Delta y = 0$

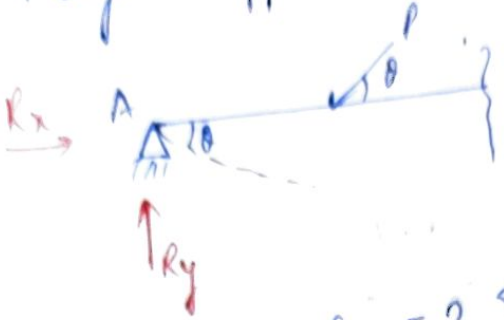
$\theta = 0$

No. of support reaction ($r_e$)

$r_e = 3 \to R_x, R_y, M_z$

$\Rightarrow$ P is the external load not ~~an~~ a reaction.

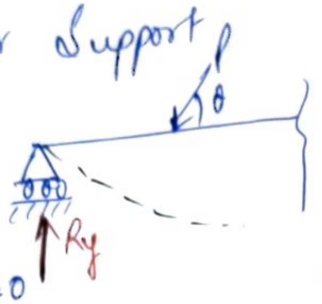
## Hinge Support




No restrain, hence moment won't be there.

$$r_e = 2 \begin{cases} \rightarrow R_x \\ \rightarrow R_y \end{cases}$$

## ③ Roller Support

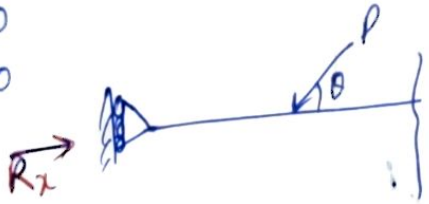



$\Delta x \neq 0$

$\Delta y = 0$

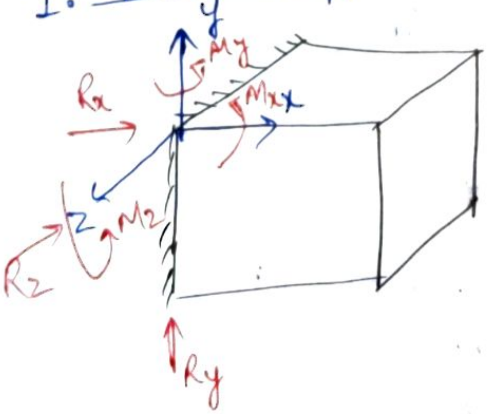
$\theta \neq 0$

$r_e = 1$




## 3-D loading / Space structure

1. Fixed Support




$r_e = 6$

$$\begin{bmatrix} R_x & M_x \\ R_y & M_y \\ R_z & M_z \end{bmatrix}$$

Bending Moment → Lateral Axis $(M_y, M_z)$

Twisting Moment → Longitudinal Axis $(M_x)$

* Hinged Support

$r_e = 3 \quad \begin{bmatrix} R_x \\ R_y \\ R_z \end{bmatrix}$

## * INTERNAL FORCES

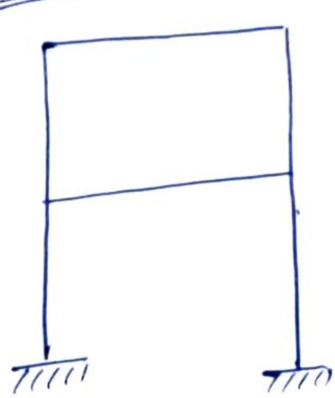
Internal Force (IF) $\longrightarrow$ $\left\{ \begin{array}{l} \text{Axial Force} \\ \text{Shear Force} \\ \text{Bending Moment} \\ \text{Twisting Moment} \end{array} \right\}$

**Structures**

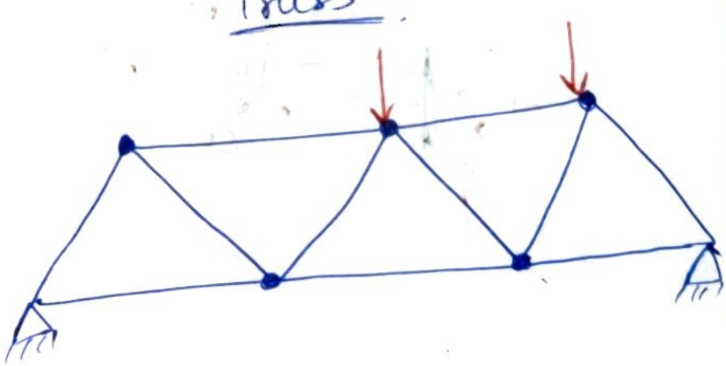
① ~~Rigid~~ Pin Jointed Structures
→ (Truss)

② Rigid Jointed Structures.
→ (Frames)

Frames



Truss



Degree of Kinematic Indeterminacy or
Degree of Freedom

$D_k = 3j - R +$ No. of internal hinges

Static & Kinematic Indeterminacy for Beams

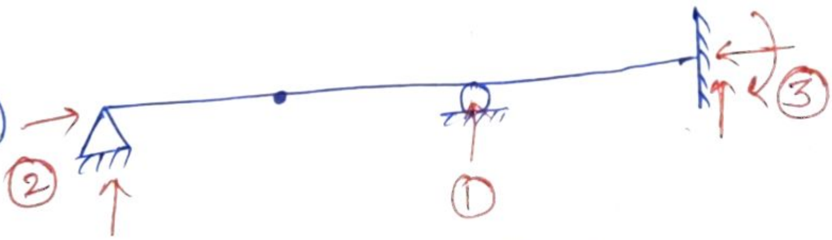
① 

$D_s = 6 - 3 - 1 = 2$

$D_s = 2$

$D_k = 3j - R + h$ $\quad j = 3$
$= 3 \times 3 - 6 + 1$ $\quad R = 6$
$= 9 - 6 + 1$ $\quad h = 1$

$D_k = 4$

② 

$D_s = (2 + 1 + 3) - 3 - 1$
$= 6 - 4 = 2$ $\quad \therefore \left[D_s = 2\right]$

$D_k = 3j - R + h$ $\quad j = 4$
$= 3 \times 4 - 6 + 1$ $\quad R = 6$
$= 12 + 1 - 6 = 7$ $\quad h = 1$

$\therefore \left[D_k = 7\right]$

* Degree of Static Indeterminacy or Degree of Redundancy

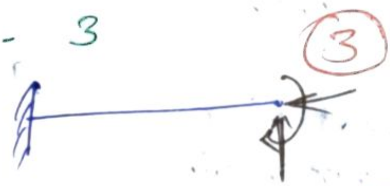
No. of equations required over and above the equations of static equilibrium for the analysis of structure is known as the degree of static Indeterminacy or degree of redundancy of the structure.

* Degree of Kinematic Indeterminacy or degree of freedom:

The number of equilibrium conditions needed to find the displacement components of all joints of the structure is known as degree of freedom

Degree of Freedom

① Free End - 3




② Roller End - 2


③ Hinged End - 1

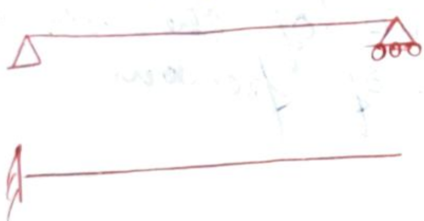

④ Fixed End - 0

## Determinate Structure

- Structure can be analyzed by using conditions of equilibrium is called as statically determinate structure

$$\Sigma F_x = 0,\ \Sigma F_y = 0,\ \Sigma M = 0$$

- No stresses due to change in temperature
- No stresses due to lack of fit.

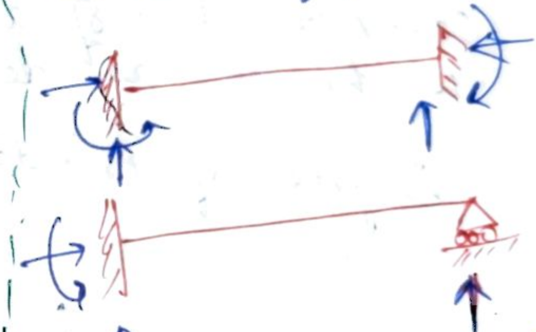


Three Hinged Arch

- c/s and material properties are not required

## Indeterminate Structure

- Structure cannot be analyzed by using conditions of equilibrium is called statically indeterminate structure
- Stresses are caused due to change in temp.
- Stresses are developed due to lack of fit



- Cross section & Material properties are required.

Degree of Static Indeterminacy or degree of redundancy

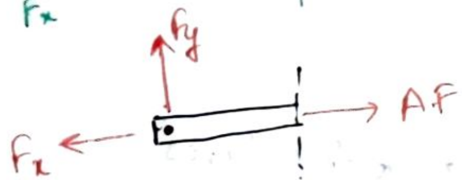
= Actual no. of unknown reactions − Available equations of equilibrium − $\boxed{\text{No. of internal Hinges}}$

## Truss

* Pin jointed "Structures"
* All members of the truss only axial force

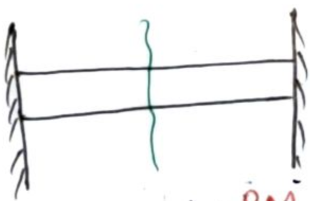
## Assumptions.

* External load are always placed on the joints

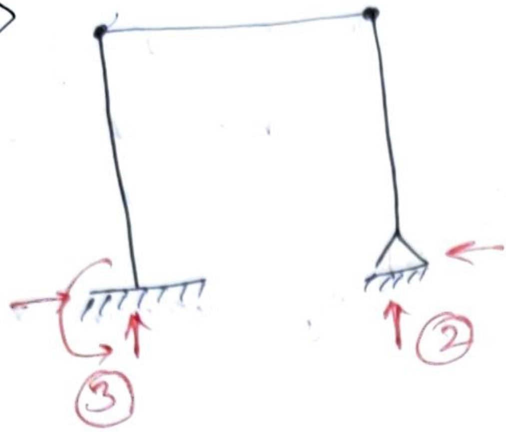

$\Sigma F = 1$

## Rigid Structure




$\Sigma F = 3$

1)




$D_s = 3m + R - 3j - h$

$m = 3, \ R = 3+2$

$j = 4, \ h = 2$

$D_s = 3\times3+5-3\times4-2$

$= 9+5-12-2 = 0$

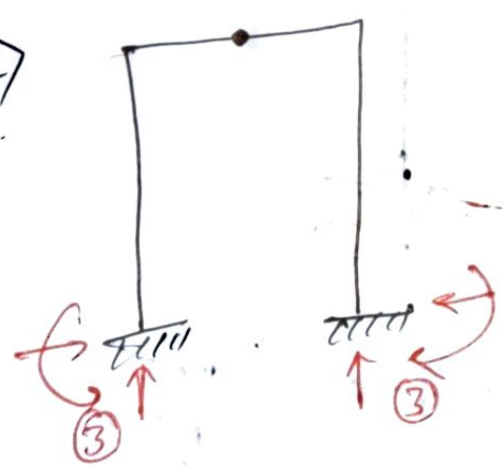
$[D_s = 0]$

$D_K = 3j - R + h$

$= 3\times4 - 5+2 = \cancel{8}\ 9$

$[D_K = 9]$

2)




$m = 4, \ R = 3+3 = 6$

$j = 5, \ h = 1$

$D_s = 3\times4 + 6 - 3\times5 - 1$

$= 18 - 16 = 2$

$[D_s = 2]$

$D_K = 3\times5 - 6+1$

$= 15 - 5$

$[D_K = 10]$

# Degree of Static & Kinematic Indeterminacy for Frames

Degree of Static Indeterminacy or degree of redundancy

$$D_s = 3m + R - 3j - h$$

m = no. of members
R = no. of unknown reactions
h = Internal hinges

Degree of Kinematic Indeterminacy or degree of freedom

$$D_k = 3j - R + h$$

| Structure | $D_{si}$ |
|---|---|
| 2D Truss | $m+r-2j$ |
| 2D Frame | $3m+r-3j-h$ |
| 3D Truss | $m+r-3j$ |
| 3D Frame | $6m+r-6j-h$ |

(5)

$D_s$ = No. of unknown reactions − Equilibrium Eqns. − I.H

= (3+2) − 3 − 2 = 0   [$D_s$ = 0]

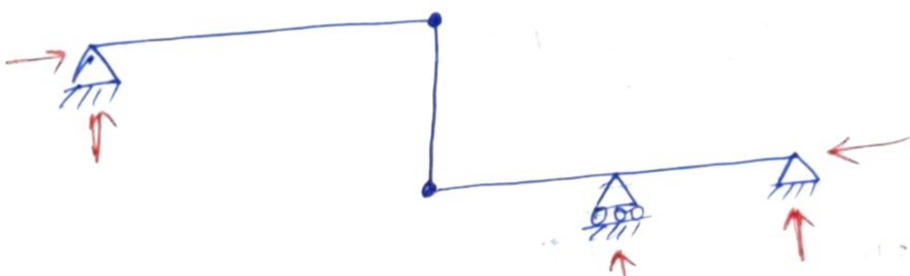
$D_k$ = 3j − R + No. of Internal Hinges

= 3× 4 − 5 + 2 = 12 + 2 − 5

= 9

[$D_k$ = 9]

(6)



$D_s$ = (2+2+1) − 3 − 2

= 5 − 5 = 0

[$D_s$ = 0] → Determinate

$D_K$ = 3j − R + h   j = 5

R = 5

h = 2

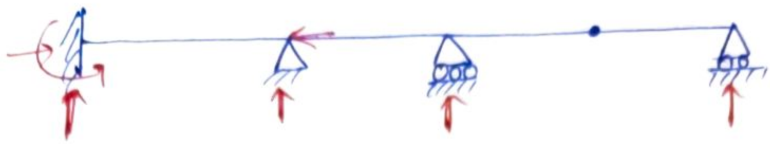
= 3×5 − 5 + 2

= 15 − 5 + 2

= 17 − 5

[$D_k$ = 12]

③



$D_s$ = No. of unknowns − Available equilibrium eq$^{ns}$ − I.H.

$= (3+2+1+1) - 3 - 1$

$D_s = 7 - 4 = 3$ $\quad [D_s = 3]$

$D_k = 3j - R + I.H.$ $\quad j = 5,\ R = 7,\ h = 1$

$= 3 \times 5 - 7 + 1$

$= 9$

$[D_k = 9]$

④

$D_s = 3 + 3 + 1 - 3 - 1$

$= 7 - 3 - 1 = 4$

$[D_s = 4]$

$D_k = 3j - R + h$ $\quad j = 4 \quad h = 1$

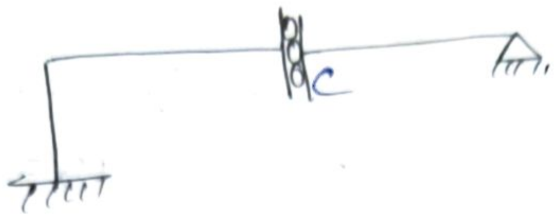
$R = 7$

$= 3 \times 4 - 7 + 1$

$= 12 - 7 + 1$

$[D_k = 6]$

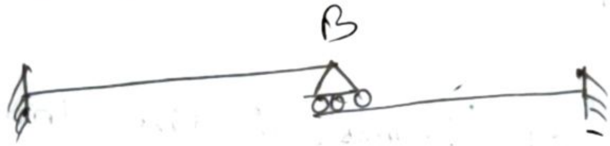
③ Internal Slider




Property SF = 0

IF = 2 (AF, BM)

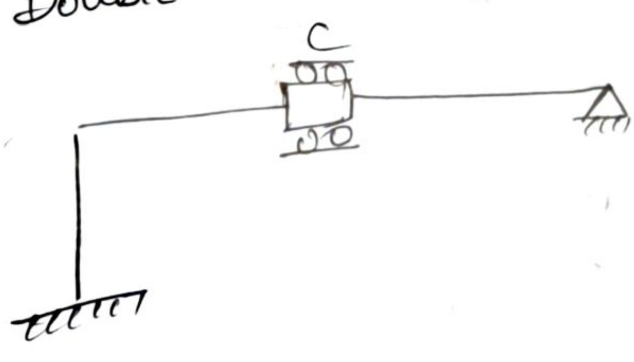
④ Internal Roller




AF = 0
BM = 0

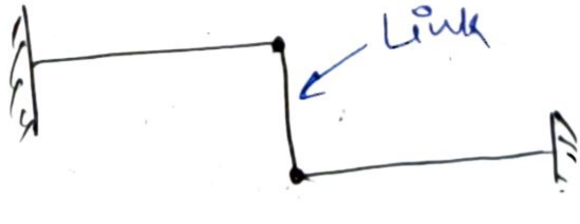
IF = 1 (SF)

⑤ Double roller




AF = 0
IF = SF, BM

⑥ Link




SF = 0
BM = 0

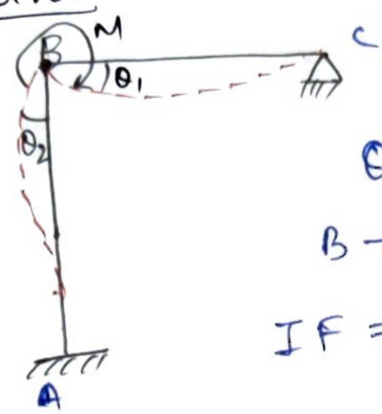
IF = 1 (AF)

## * Indeterminate Structures

→ cannot be analysed using st. equil. equations. It requires additional comp. equations in the form of slope & deflection equations.

### INTERNAL JOINTS

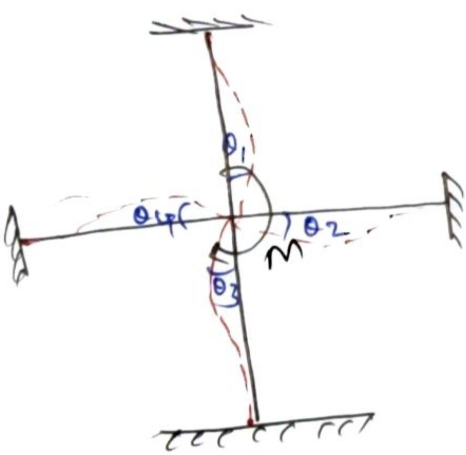
① Rigid Joints




$\theta_1 = \theta_2$

B → rigid joint

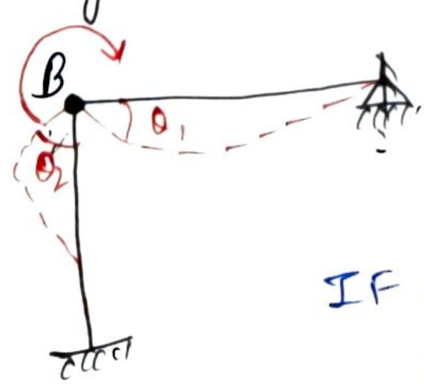
IF = AF, SF, BM } 3




$\theta_1 = \theta_2 = \theta_3 = \theta_4$

② Internal Hinge

$\theta_1 = \theta_2$




At B,

$\boxed{BM = 0}$

IF = 2

= AF, SF

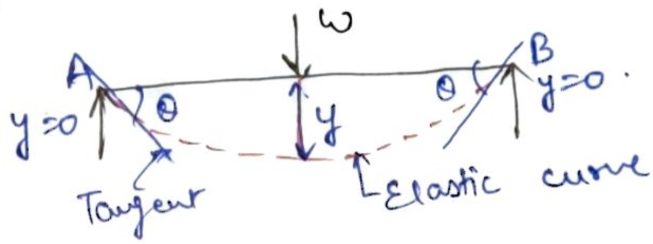
# Deflection in a Beam

→ The amount of beam deflection depends on the size of the beam, the materials used, and the weight and position of any object placed on it.

Deflection: It is the vertical distance of the beam measured before & after loading.

→ It is denoted by 'y'

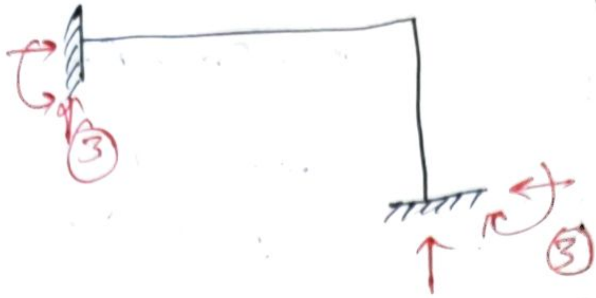
→ Deflection at supports is always zero.




Slope: It is the angle measured in radians measured b/w the tangent to the elastic curve & the original axis of the beam

→ Slope is denoted by 'θ' or '$\frac{dy}{dx}$'

→ Its unit is radians.

3) 

$D_s = 3m + R - 3j - h$

$m = 2,\ R = 6$

$j = 3$

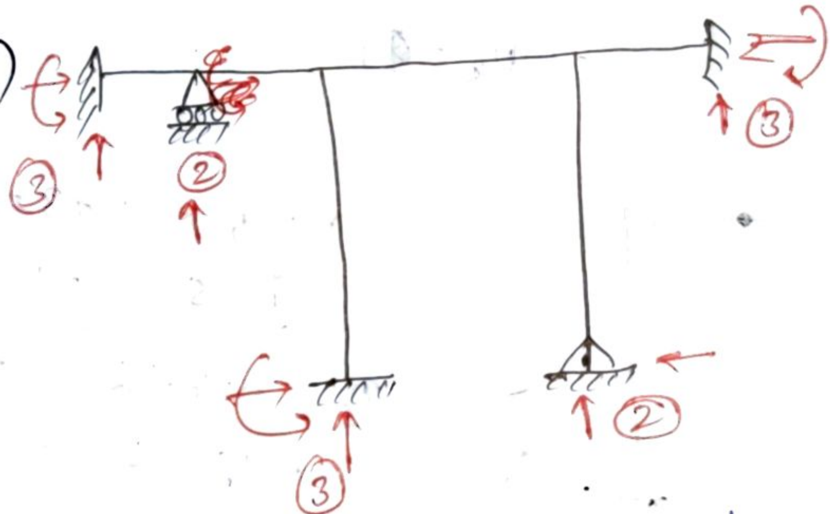
$D_s = 3\times2 + 6 - 3\times3$

$= 3$

$[D_s = 3]$

$D_k = 3j - R + h \quad = 3\times3 - 6 + 0$

$[D_k = 3]$

4) 

$D_s = 3m + R - 3j - h$

$m = 6$
$R = 12$
$j = 7$

$D_s = 3\times6 + 12 - 3\times7$

→ concept of general loading and vertical loading
In case of beams, if the loading is vertical and the supports are at same level, then no horizontal reaction will develop.

⇒ Methods of analysis of Indeterminate structures.

① Exact Method. (small structures)
(i) Force Method
(ii) Displacement Method.

② Approximate Method (complex ~~met~~ structures)
(i) Cantilever
ii) Portal Frame Method.

| Force Method (compatibility) | Displacement Method (Equilibrium) |
|---|---|
| → Member forces & external supp. reactions. ↳ Unknown | → Joint displacement ($\theta, \Delta$) → unknown |
| → Compatibility eq$^n$ are written equal to the value of ($D_s$) | → They are written equal to the value of ($D_k$) |
| $D_s < D_k$ | $D_k < D_s$ |
| → Consistent Deformation<br>→ Strain Energy Method<br>→ Three Moment Theorem | → Moment Dist. Method<br>→ Slop & Deflection<br>→ Kani's Method<br>→ Matrix Method. |

7

# Indeterminacy

| Static Indeterminacy | Kinematic Indeterminacy |
| --- | --- |
| → Based on external support reactions & their internal member forces or geometry | → Based on their degree of freedom available at all joints. |

* Static Indeterminacy

① External Static Indeterminacy

$(D_{se}) = r_e - E$

② Internal Static Indeterminacy.

- Truss → 2D, → 3D
- Frames, Beams → 2D, → 3D

Truss : It is called as pin jointed frame / structure. Following are the assumptions for trusses:

→ Hook's law is valid.

→ All members are perfectly straight.

→ All members are exerting only axial forces.

→ In case of beams, if the loading is vertical but supports are not at same level; then the horizontal reactions will be considered.

→ If in case of frames or truss, horizontal loading is considered in every situation.

* Types of Structure on their determinacy.

① Determinate Structure

→ If all the support reactions & member forces can be found out using equi$^{n}$ eq$^{n}$.

Properties

→ To analyse, material properties & cross-sectional properties are not required.

Material Properties : (E, G)

Cross-sectional properties : (A, I)

② Indeterminate Structure

→ If all the support reactions cannot be found out using equilibrium equation, then the structure will be called as indeterminate structure

To analyse, we need additional equations known as compatibility equations in the form of slope and deflection equation; equations (E, G) & (A, I) are required.

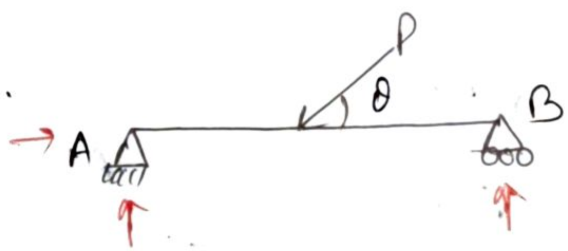
# Equilibrium Equation (E)

| 2D (Total : 3) | 3D (Total : 6) | |
|---|---|---|
| $\Sigma F_x = 0$ | $\Sigma F_x = 0$ | $\Sigma M_x = 0$ |
| $\Sigma F_y = 0$ | $\Sigma F_y = 0$ | $\Sigma M_y = 0$ |
| $\Sigma M = 0$ | $\Sigma F_z = 0$ | $\Sigma M_z = 0$ |

* Concept of General loading and vertical loading

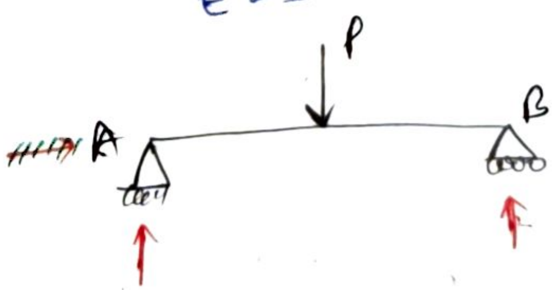
→ In case of 'beams', if the load is 'vertical' and, the supports are at 'same level': then no horizontal reaction will develop.

eg.




$r_e = 3$

$E = 3$




$r_e = 2$

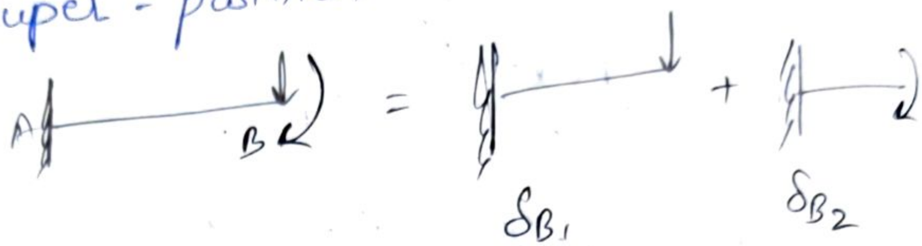
$E = 2$

$F_x$ & $\Sigma F_x$ won't be there.

$E, I \Rightarrow$ Crosssectional & Material Properties.

## Theorems

1. Super-position Theorem

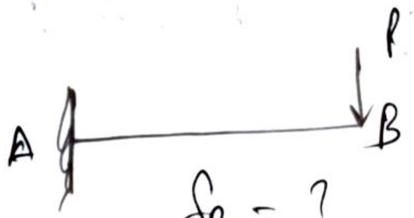


$$\delta_B = \delta_{B_1} + \delta_{B_2}.$$

2. Castigliano's Theorem

$I^{st}$

$$\frac{\partial U}{\partial P} = \delta \qquad \frac{\partial U}{\partial M} = \theta.$$



$\delta_B = ?$

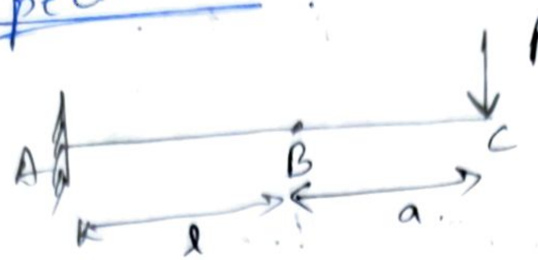
$U_B =$

$$\frac{\partial U}{\partial P} = \delta_B.$$

$II^{nd}$.

$$\boxed{\frac{\partial U}{\partial \delta} = P} \qquad \boxed{\frac{\partial U}{\partial \theta} = M}$$

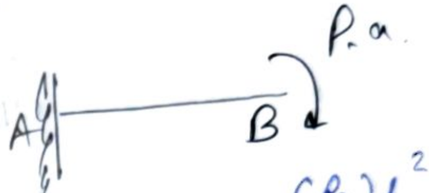
## Special Cases




$$\delta_c = \frac{P(l+a)^3}{3EI}, \quad \delta_B = ?$$

### Super-position theorem

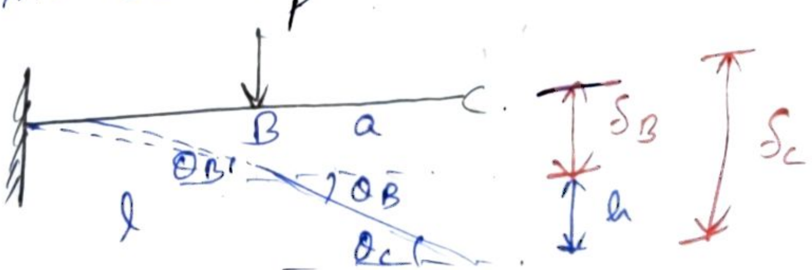

&




$$\delta_{B_1} = \frac{Pl^3}{3EI} \qquad \delta_{B_2} = \frac{Ml^2}{2EI} = \frac{(Pa)l^2}{2EI}$$

$$\delta_B = \delta_{B_1} + \delta_{B_2} = \frac{Pl^3}{3EI} + \frac{(Pa)l^2}{2EI}$$

** Find $\delta_c = ?$




$$\theta_B = \theta_c = \frac{Pl^2}{2EI}$$

$$\delta_c = \delta_B + h = \frac{Pl^3}{3EI} + h$$

$\underline{h}$

$$\tan\theta_B = \frac{h}{a} \simeq \theta_B = \frac{h}{a}$$

$$h = \theta_B \times a$$

$$\boxed{\delta_c = \frac{Pl^3}{3EI} + \theta_B \times a}$$

(8)

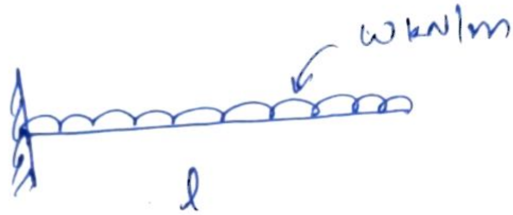
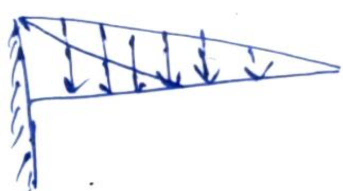

$$\theta_A = \frac{Ml}{6EI}$$

$$\theta_B = \frac{Ml}{3EI}$$

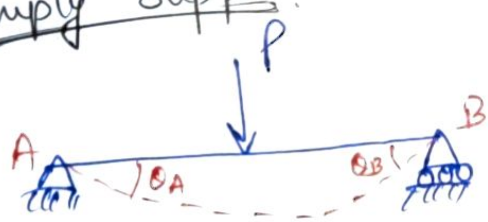
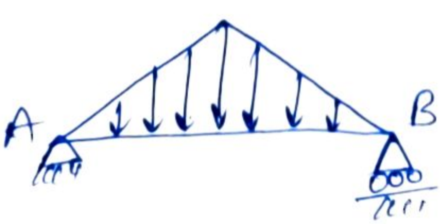
(9)

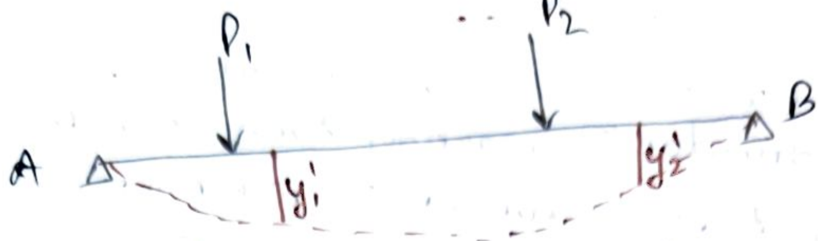

$$\theta_A = \theta_B = \frac{Ml}{2EI}$$

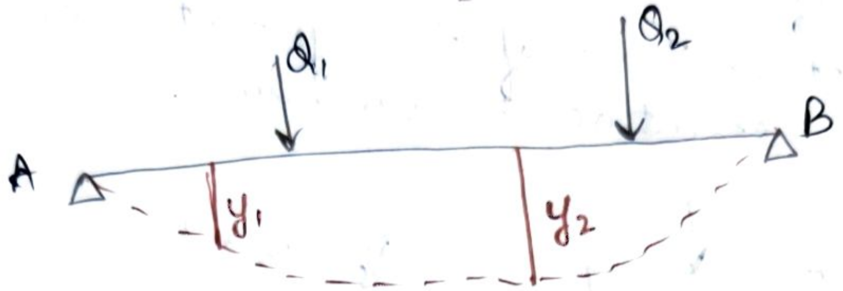
## Cantilever

| | $\theta$ | $\delta$ |
|---|---|---|
| ①  | $\theta_B = \frac{Pl^2}{2EI}$ | $\delta_B = \frac{Pl^3}{3EI}$ |
| ②  | $\theta_B = \frac{Ml}{EI}$ | $\delta_B = \frac{Ml^2}{2EI}$ |
| ③  | $\theta_B = \frac{wl^3}{6EI}$ | $\delta_B = \frac{wl^4}{8EI}$ |
| ④  | $\theta_B = \frac{wl^3}{24EI}$ | $\delta_B = \frac{wl^4}{30EI}$ |

## Simply Supported

| | $\theta$ | $\delta$ |
|---|---|---|
| ⑤  | $\theta_A = \theta_B = \frac{Pl^2}{16EI}$ | $\delta_{max} = \frac{Pl^3}{48EI}$ |
| ⑥  | $\theta_A = \theta_B = \frac{wl^3}{24EI}$ | $\delta_{max} = \frac{5wl^4}{384EI}$ |
| ⑦  | $\theta_A = \theta_B = \frac{5wl^3}{192EI}$ | $\delta_c = \frac{wl^4}{120EI}$ |



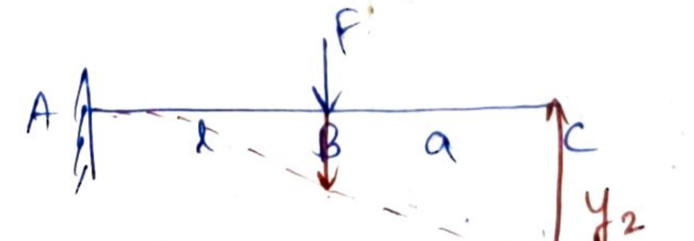

$$W_{P\text{-system}} = P_1 y_1 + P_2 y_2$$

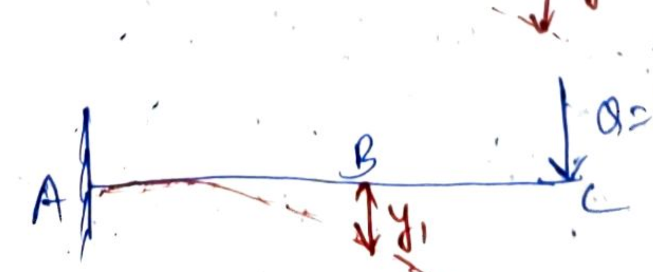
$$W_{Q\text{-system}} = Q_1 y_1' + Q_2 y_2'$$

$$W_{P\text{-system}} = W_{Q\text{-system}}$$

$$\boxed{P_1 y_1 + P_2 y_2 = Q_1 y_1' + Q_2 y_2'}$$

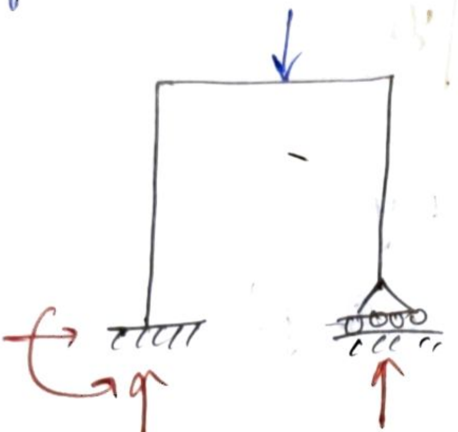
e.g.:




IV ** Minimum Potential Energy Theorem

If a structure is loaded and there is any redundant reaction then the true value of that redundant reaction can be determined if the potential energy of the structure is minimum.



$D_s = r_e - E$
$= 4 - 3 = 1$

$U \Rightarrow$ Minimum

This theorem states that the partial derivative of the potential energy w.r.t redundant reaction is 0.

$U \longrightarrow$ Minimum

$$\boxed{\frac{\partial U}{\partial R} = 0}$$

$R$ = redundant reaction

V Betti's Law (Rayleigh Theorem)

[material - linear elastic
Temp - constant
supports - Unyeilding]

According to betti's law the virtual work done by P-system of force in going through displacement caused by Q-system of forces is equal to the virtual work done by Q-system of forces in going through displacement by P-system of forces.

$$\boxed{\frac{\partial U}{\partial \delta} = P}$$

$$\boxed{\frac{\partial U}{\partial \theta} = M}$$

## III Castigliano's 2nd Theorem

- Temperature - constant
- Linearly Elastic
- Supports unyielding

Then the first partial derivative of total strain energy w.r.t load/moment will give the deflection/slope respectively.

$$\boxed{\frac{\partial U}{\partial P} = \Delta \text{ or } \delta}$$

$$\boxed{\frac{\partial U}{\partial M} = \theta}$$

This theorem is also called as Strain Energy Method.

## I Super Position Theorem

→ According to this theorem, the resultant stress function for multiple loading is equals to the sum of the effects of individual loading

Stress function : Slope, Deflection etc.

→ It is valid for beams & frames for both determinate & Indeterminate structures.

**Assumptions** : * Material → Isotropic, → Homogeneous, → Linear Elastic

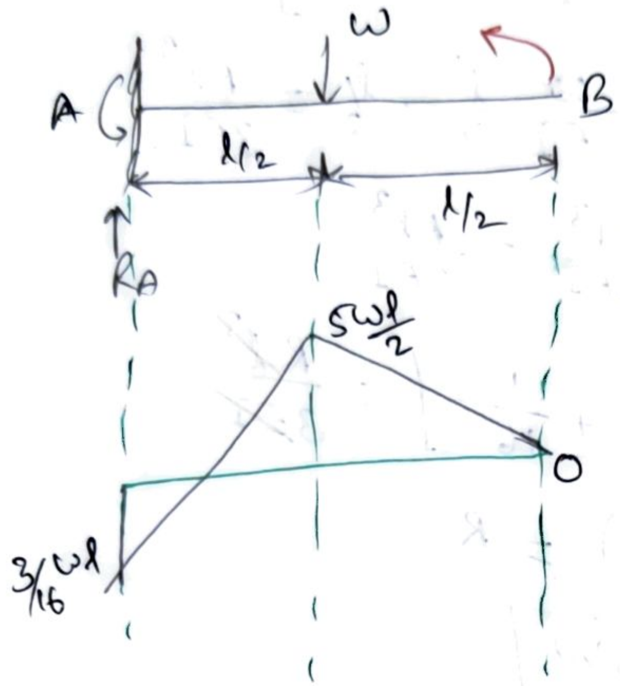
Hook's law valid

* Temperature is constant
* Supports are unyeilding (don't sink)

## II Castigliano's I Theorem :

For beams, truss and frames, if the material is linear elastic, supports are unyeilding & temp. is constant, then the first partial derivative of total strain energy w.r.t to the load/moment will give the deflection/slope at that point.

Now, for B.M.D (↺ −ve ↻ +ve)




$(B.M)_B = 0$

$(B.M.)_C = \frac{5w}{16} \cdot l/_2$

$= \frac{5wl}{32}$

$(B.M.)_A = Rl - \frac{wl}{2}$

$= \frac{5w}{16} - \frac{wl}{2}$

$= wl\left[\frac{5}{16} - \frac{1}{2}\right]$

$= -\frac{3}{16} wl$

$$\therefore \left[\frac{W}{3EI}\left(\frac{l}{2}\right)^3 + \frac{W\left(l/2\right)^2}{2EI} \times l/2\right] = \frac{Rl^3}{3EI}$$

$$\therefore \frac{W}{3EI}\cdot\frac{l^3}{8} + \frac{W}{2EI}\frac{l^2}{4}\cdot\frac{l}{2} = \frac{Rl^3}{3EI}$$

$$\therefore \frac{Wl^3}{24EI} + \frac{Wl^3}{16EI} = \frac{Rl^3}{3EI}$$

$$\therefore \frac{W\cancel{l^3}}{\cancel{EI}}\left[\frac{1}{24}+\frac{1}{16}\right] = \frac{R\cancel{l^3}}{3\cancel{EI}}$$

$$3W\left(\frac{5}{48}\right) = R$$

$$\boxed{\therefore R = \frac{5W}{16}}$$

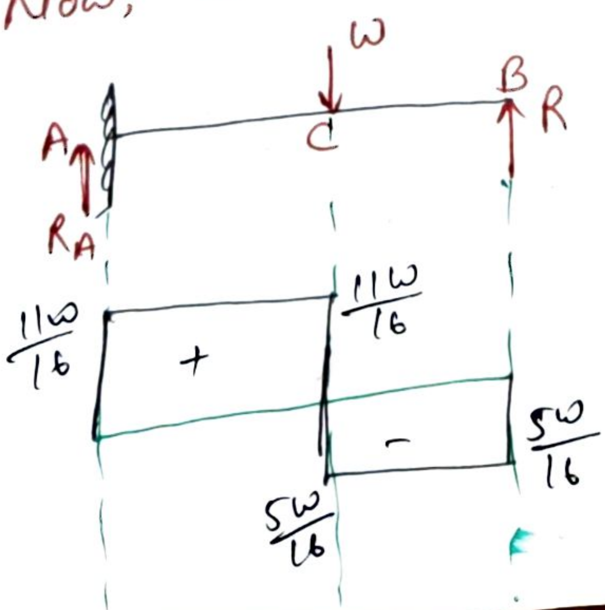
$\Sigma F_y = 0$ ($\uparrow$ +ve, $\downarrow$ -ve)

$$\therefore R_A + R - W = 0$$

$$R_A = W - R = W - \frac{5W}{16}$$

$$= \frac{11W}{16}$$

Now, S.F.D. ($\uparrow$ -ve, $\downarrow$ +ve)



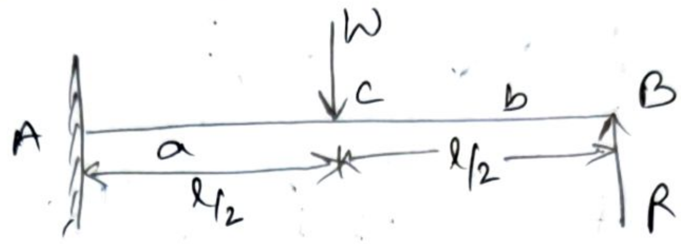

$$(S.F.)_B = R = -\frac{5W}{16}$$

$$(S.F.)_C = \frac{-5W}{16} + W$$

$$= \frac{11W}{16}$$

$$(S.F.)_A = \frac{11W}{16}$$

A cantilever of length $l$ carries a concentrated load $W$ at its midspan. If the free end be supported on rigid prop, find the reaction at the prop. Draw SFD and BMD.




→ Downward deflection at point B due to load W is given by

$$= \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI} b$$

Upward deflection at point B due to reaction R is given by.

$$= \frac{Rl^3}{3EI}$$

→ Deflection at B = 0.

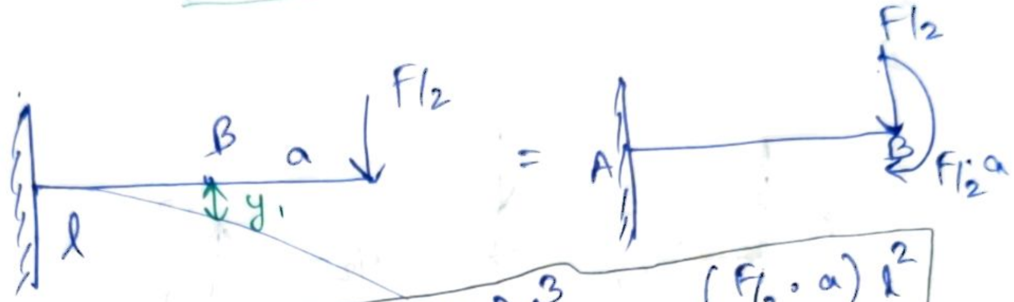
$\therefore$ {Downward Deflection of B due to load W}

= {Upward deflection of B due to R}

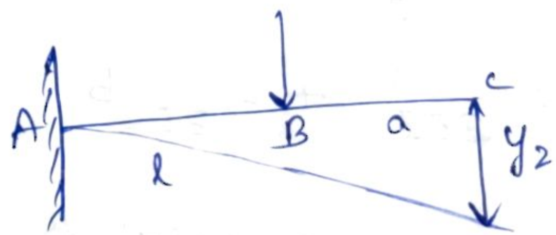
$$\therefore \left\{ \frac{Wa^3}{3EI} + \frac{Wa^2}{2EI} \cdot b \right\} = \frac{Rl^3}{3EI}$$

$W_{P\text{-system}} = W_{Q\text{-system}}$

$\boxed{P_1 y_1 = P_2 y_2} \rightarrow$ Prove




$$\boxed{y_1 = \frac{(F/2)\, l^3}{3EI} + \frac{(F/2 \cdot a)\, l^2}{2EI}}$$




$$y_2 = y_B + \theta_B \cdot a$$

$$\boxed{y_2 = \frac{F l^3}{3EI} + \frac{F l^2}{2EI} \cdot a}$$

## Special case-I. Betti's law

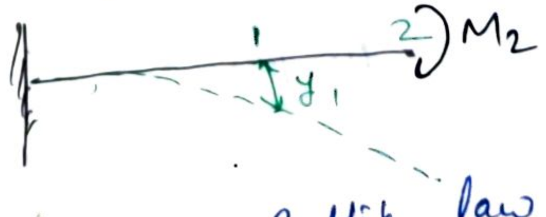
* If $\theta_1$ is the rotational displacement in the direction of $M_1$ caused by $M_2$ and $\theta_2$ be the rotational displacement in the direction of $M_2$ caused by $M_1$.



Betti's law

$$M_1\theta_1 = M_2\theta_2$$

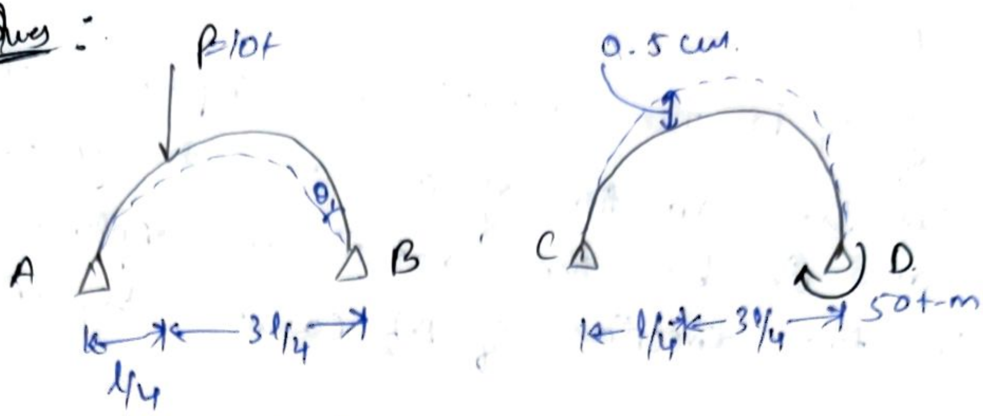
* Special Case:


As per Betti's law

$$\boxed{P_1 y_1 = P_2 y_2}$$

Ques :




$\theta = ?$

According to Betti's law

$$P y_1 = M \theta_1$$

$$P y_1 = 10t \times 0.5$$

$$M\theta_1 = 50t \times \theta_1$$

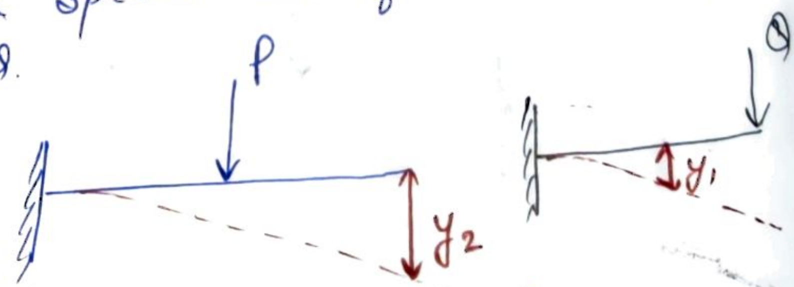
$$10t \times 0.5 = 50t \times \theta_1$$

$$\theta_1 = 0.1 \text{ radians}$$

Maxwell Reciprocal Theorem

It is a special case of Betti's law, here $P = Q$.

e.g.




As per Betti's law

$$P y_1 = Q y_2$$

if $P = Q$

$$\boxed{y_1 = y_2}$$

"Method of Consistent Deformation"

* Procedure

① Find $D_s$

② Remove the redundation reaction and find out the displacement ($\theta$ or $\Delta$) in the direction of the redundant reaction.

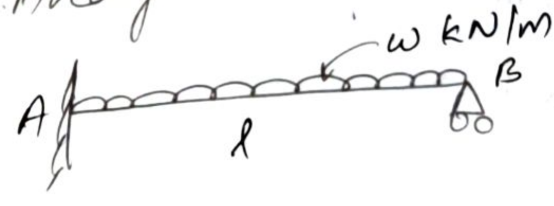
Force → Deflection ($\delta$)

Moment → Slope ($\theta$)

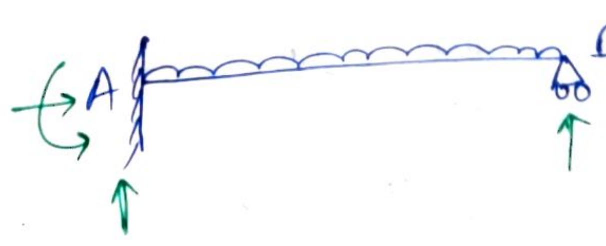
③ Remove the applied loading and find out the displacement in the direction of the loading.

④ Obtain compatibility equation & find out the redundant reaction

Ques Analyse the structure




Soln :




Step 1 :

$$D_s = r_e - E - h$$
$$= 4 - 3 - 0 = 1$$

**Ques**


**Sol** (1)

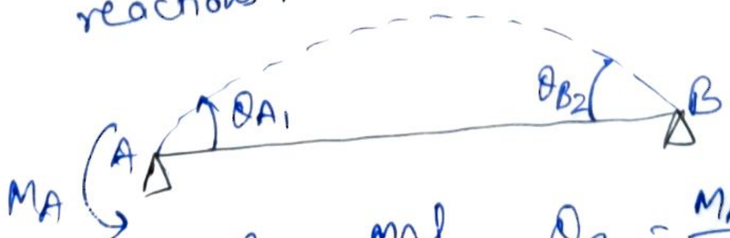
$$D_S = r_e - E$$
$$= 4 - 2 = 2.$$

$[M_A, M_B] \rightarrow$ redundant

(2) Remove the redundant reaction and find the displacement due to loading.


$$\theta_{A_1} = \frac{wl^3}{24EI}, \quad \theta_{B_1} = \frac{wl^3}{24EI} \ (\circlearrowright)$$

(3) Remove the applied loading & find out the displacement due to redundant reactions.




$$\theta_{A_2} = \frac{M_A l}{3EI} \qquad \theta_{B_2} = \frac{M_A l}{6EI}$$

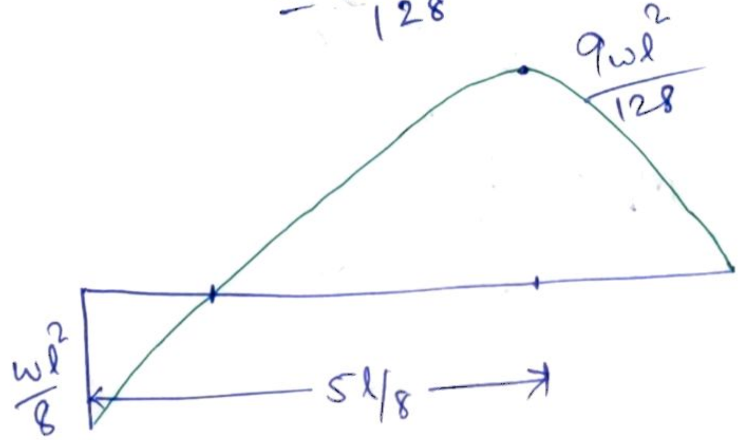
$$\frac{dM_x}{dx} = \frac{5wl}{8} - wx = 0$$

$$\cancel{w}x = \frac{5\cancel{w}l}{8}$$

$$\boxed{x = \frac{5l}{8}} \rightarrow \text{from A}$$

$$M_{x-x} = \frac{5wl}{8} \times \frac{5l}{8} - \frac{wl^2}{8} - w\left(\frac{5l}{8}\right)^2$$

$$= \frac{9wl^2}{128}$$




Put ~~xxx~~ $M_{x-x} = 0$ in the equation 1 to calculate the point of contraflexure.

$R_A + R_B = \omega l$

$R_A = \omega l - \frac{3\omega l}{8}$  $\therefore R_A = \frac{5\omega l}{8}$

~~$R_B \times l - \frac{\omega l \times l}{2} = 0$~~

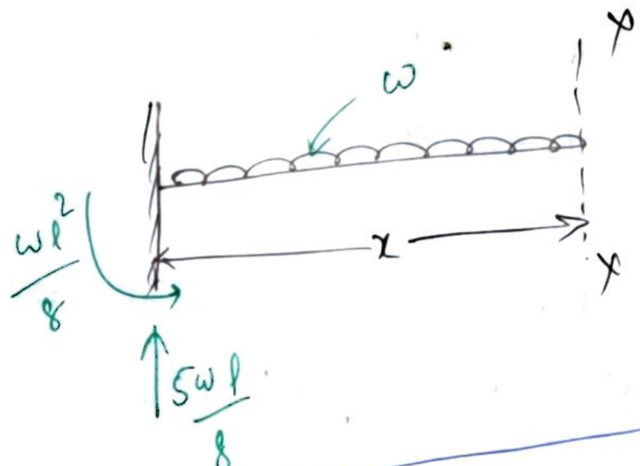
$\Sigma M = 0$

$R_B - \frac{\omega l^2}{2} + M_A = 0$

$\therefore M_A = \frac{\omega l^2}{2} - R_B$

$= \frac{\omega l^2}{2} - \frac{3\omega l}{8} \times l$

$M_A = \frac{\omega l^2}{8}$




$$\boxed{M_{xx} = \frac{5\omega l}{8} \cdot x - \frac{\omega l^2}{8} - \frac{\omega x^2}{2}} \text{ --- } (1)$$

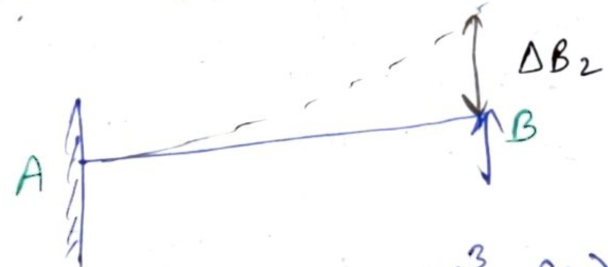
$\frac{dM_{x-x}}{dx} = 0$ (Maxima)

Let $R_B \to$ redundant reaction

Step 2 : Remove $R_B$ and find $\Delta_{B_1}$

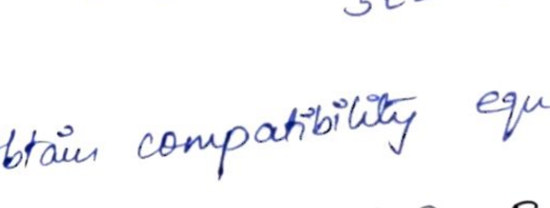

$$\Delta B_1 = \frac{w l^4}{8EI} (\downarrow)$$

Step 3 : Remove the loading, apply redundant reaction & find the displacement ($\Delta B_2$)




$$\Delta B_2 = -\frac{R_B l^3}{3EI} (\uparrow)$$

Step 4 : Obtain compatibility equation




$$\boxed{\Delta B = 0} \to \text{compatibility equation.}$$

$$\frac{w l^4}{8EI} - \frac{R_B l^3}{3EI} = 0$$

$$\frac{w l^4}{8EI} = \frac{R_B l^3}{3EI}$$

$$\therefore R_B = \frac{3wl}{8}$$


$$\theta_{A_3} = \frac{M_B l}{6EI} \qquad \theta_{B_3} = \frac{M_B l}{3EI}.$$

④ Compatibility Equation:

since A, B are fixed supports

$$\left.\begin{array}{l}\theta_A = 0 \\ \theta_B = 0\end{array}\right\} \rightarrow \text{C.E.}$$

$$\theta_A = 0$$

$$\theta_{A_1} + \theta_{A_2} + \theta_{A_3} = 0$$

$$\frac{wl^3}{24EI} - \frac{M_A l}{3EI} - \frac{M_B l}{6EI} = 0$$

$$\theta_B = 0$$

$$\theta_{B_1} + \theta_{B_2} + \theta_{B_3} = 0$$

$$\frac{-wl^3}{24EI} + \frac{M_A l}{6EI} + \frac{M_B l}{3EI} = 0$$

$$U = \int \frac{P^2 \cdot dx}{2AE}$$ → Axial load (Truss)

$$U = \int \frac{M_x^2 \cdot dx}{2EI}$$ → B.M (Frames, Beams)

$$U = \int \frac{T^2 \cdot dx}{2GJ}$$ → T.M

## Castigliano's Theorem

$$\frac{\partial U}{\partial P} = \Delta \quad ; \quad \frac{\partial U}{\partial M} = \theta$$

$$\Delta = \frac{\partial U}{\partial P} = \frac{\partial}{\partial P} \int \frac{M_x^2 \, dx}{2EI}$$

$$= \int \frac{\cancel{2} M_x \cdot \frac{\partial M_x}{\partial P} \, dx}{\cancel{2} EI}$$

$$\boxed{\Delta = \int_0^l \frac{M_x \frac{\partial M_x}{\partial P} \, dx}{EI}}$$

$$\boxed{U = \frac{1}{2} P \times \frac{PL}{AE} = \frac{P^2 L}{2AE}}$$

Non-uniform Body


$$\boxed{U = \int_0^L \frac{P^2}{2AE}\, dx}$$

## BENDING MOMENT

$$U = \frac{\sigma_b^2}{2E} \times \text{Volume}$$

$$= \int \frac{\left(\frac{My}{I}\right)^2}{2E} \times dA \times dx$$

$$= \int \frac{M^2 y^2}{2EI^2} \cdot dA \cdot dx$$

Bending stress

$$\frac{M}{I} = \frac{E}{R} = \frac{\sigma_b}{y}$$

$$M/I = \sigma_b / y$$

$$\sigma_b = \frac{My}{I}$$

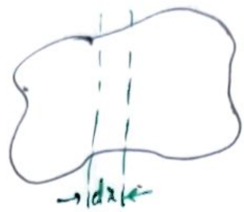
$\int y^2 \cdot dA$ = second moment of Area $= I$

$$U = \int \frac{M_x^2 \cdot \cancel{I} \cdot dx}{2EI^{\cancel{2}}}$$

$$\boxed{U = \int \frac{M_x^2\, dx}{2EI}}$$

$$\boxed{U = \tfrac{1}{2} P \times \frac{PL}{AE} = \frac{P^2 L}{2AE}}$$

Non-uniform Body



$$\boxed{U = \int_0^L \frac{P^2}{2AE}\, dx}$$

## BENDING MOMENT

$$U = \frac{\sigma_b^2}{2E} \times \text{Volume}$$

$$= \int \frac{\left(\frac{My}{I}\right)^2}{2E} \times dA \times dx$$

$$= \int \frac{M^2 y^2}{2EI^2} \cdot dA \cdot dx$$

Bending stress

$$\frac{M}{I} = \frac{E}{R} = \frac{\sigma_b}{y}$$

$$M/I = \sigma_b/y$$

$$\sigma_b = \frac{My}{I}$$

$\int y^2 \cdot dA$ = second moment of Area $= I$

$$U = \int \frac{M_x^2 \cdot \cancel{I} \cdot dx}{2EI^{\cancel{2}}}$$

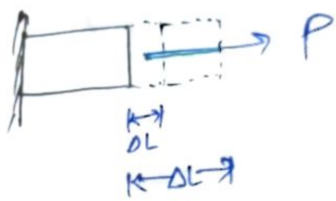
$$\boxed{U = \int \frac{M_x^2\, dx}{2EI}}$$

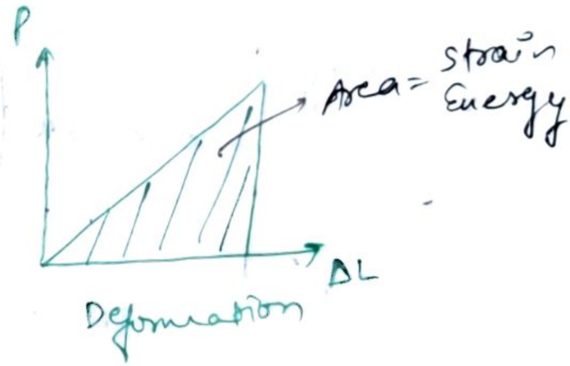
Method of Strain Energy

When a gradual load is applied on an elastic body, the body is strained and work is done on the body which is stored in the form of internal energy. This internal work done by the body in resisting the straining is called strain energy.

AXIAL LOAD

Strain Energy
↳ Internal work done to resist the straining




$$\boxed{U = \frac{1}{2} P \times \Delta L}$$

Hook's law

$\sigma \propto \epsilon$

$\sigma = E\epsilon$

$\epsilon = \frac{\sigma}{E}$

$U = \frac{1}{2} P \cdot \Delta L$

$= \frac{1}{2} (\sigma A) \cdot \epsilon L$

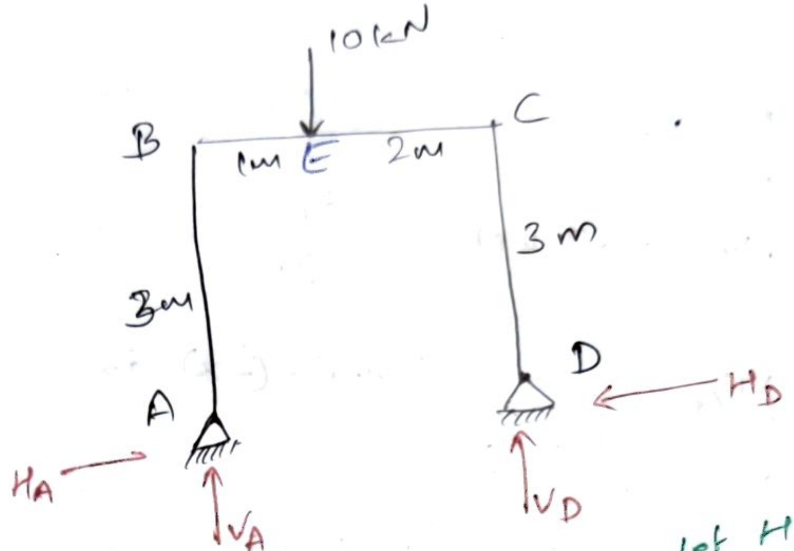
$= \frac{\sigma \times \sigma}{2E} \times AL$

$$\boxed{U = \frac{\sigma^2}{2E} \times \text{Volume}}$$

$\frac{1}{2} \sigma^2 \cdot \frac{PL}{E}$ AE

$$\boxed{\frac{\partial U}{\partial R_1} = 0 \qquad \frac{\partial U}{\partial R_2} = 0}$$

Ques




Solⁿ :

$D_s = r_e - E = 4 - 3 = \underline{1}$ let $H$ be the redundant reaction.

$\Sigma f_x = 0$

$H_A = H_D = H$

$\Sigma f_y = 0$

$V_A + V_D = 10\ kN$

$\Sigma M_A = 0$

$10 \times 1 - V_D \times 3 = 0$

$V_D = 10/3 = 3.33\ kN$

$V_A = 6.67\ kN$

$$\frac{\partial U}{\partial H} = 0$$ (from minimum strain energy theorem)

$$\boxed{\frac{\partial U_{AB}}{\partial H} + \frac{\partial U_{BE}}{\partial H} + \frac{\partial U_{EC}}{\partial H} + \frac{\partial U_{CD}}{\partial H} = 0}$$

$$\delta_{c_2} = \int_0^h \frac{(Pl)\,l}{EI} \cdot dx = \frac{Pl^2}{EI} \int_0^h dx$$

$$\boxed{\delta_{c_2} = \frac{Pl^2}{EI} \cdot h}$$

$$\delta_c = \delta_{c_1} + \delta_{c_2}$$

$$\boxed{\delta_c = \frac{Pl^3}{3EI} + \frac{Pl^2}{EI} \cdot h}$$

$$U = \tfrac{1}{2}\, P \times \delta$$

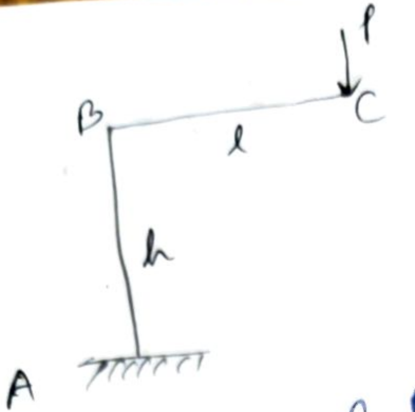
$$= \tfrac{1}{2} \times P \left[ \frac{Pl^3}{3EI} + \frac{Pl^2}{EI} \cdot h \right]$$

$$\boxed{U_c = \frac{P^2 l^3}{6EI} + \frac{P^2 l^2 h}{2EI}}$$

## Minimum Strain Energy

If $U$ is the strain energy in an elastic body & if $R_1$ and $R_2$ etc. are the redundant reactions, then if there is no support movements and no change in temperature, the redundant $R_1, R_2$ etc. must be such that the strain energy is minimum.

Ans




$\delta_c = ?$ $\quad \theta_c = ?$

$$\delta = \int \frac{M_x \frac{\partial M_x}{\partial P} \cdot dx}{EI}$$

Span BC


$$\delta_{c_{BC}} = \int \frac{M_x \frac{\partial M_x}{\partial P} dx}{EI}$$

$M_x = -Px$

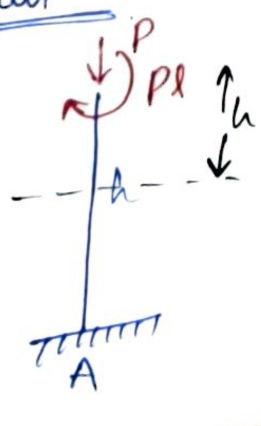
$\frac{\partial M_x}{\partial P} = -x$

$$\delta_{c_1} = \int_0^l \frac{(-Px)(-x)\, dx}{EI}$$

$$= \frac{P}{EI} \int_0^l x^2 dx$$

$$\boxed{\delta_{c_1} = \frac{Pl^3}{3EI}}$$

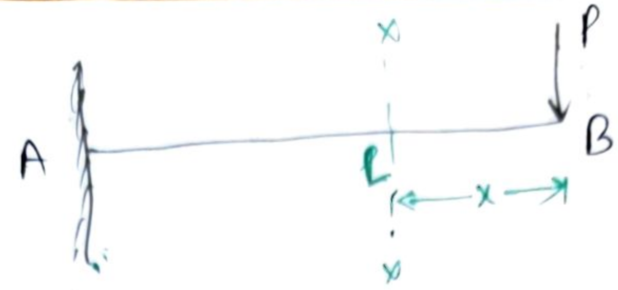
Span AB




$M_x = Pl$

$\frac{\partial M_x}{\partial P} = l$

Ques




$$\delta_B = ?$$

$$M_x = -P_x$$

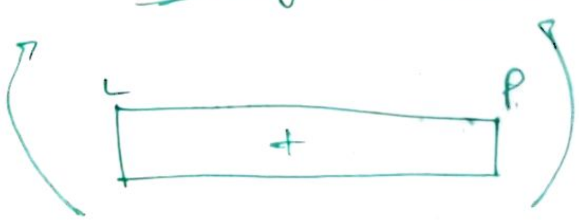
$$U = \int \frac{M_x^2 \, dx}{2EI} = \int_0^l \frac{P^2 x^2 \, dx}{2EI}$$

$$= \frac{P^2}{2EI} \int_0^l x^2 \, dx = \frac{P^2 l^3}{6EI}$$

$$U = \frac{P^2 l^3}{6EI}$$

$$\frac{\partial U}{\partial P} = \frac{2Pl^3}{6EI} = \frac{Pl^3}{3EI}$$

Bending Moment



$$\int_0^l \frac{(R_B \cdot x - P_a - P_x) \cdot x \, dx}{EI} = 0$$

$$\int_0^l \frac{(R_B x^2 - P_a \cdot x - Px^2) \, dx}{EI} = 0$$

$$\left[ R_B \frac{x^3}{3} - \frac{Pa\,x^2}{2} - \frac{Px^3}{3} \right]_0^l = 0$$

$$\frac{R_B l^3}{3} - \frac{Pa\,l^2}{2} - \frac{Pl^3}{3} = 0$$

$$R_B = P + \frac{3aP}{2l}$$

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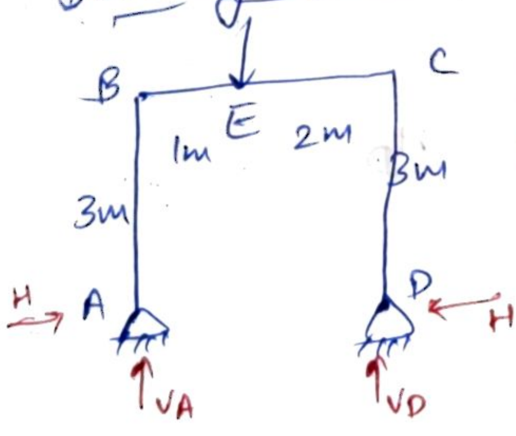
Bending Moment Diagram



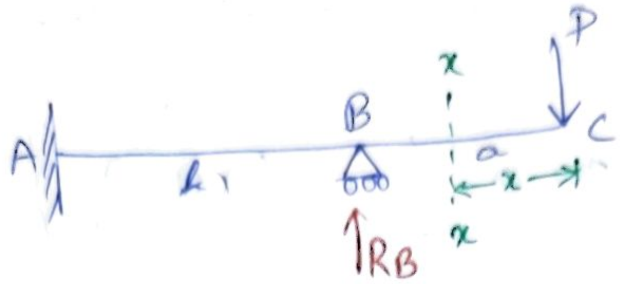

Fixed / Free

$$\begin{cases} BM_A = 0 \\ BM_B = -3H = -3 \times 0.67 = -2\,kNm \\ BM_C = -3H = -3 \times 0.67 = -2\,kNm \\ BM_D = 0 \end{cases}$$

BMD = fixed BMD + free BMD

External load

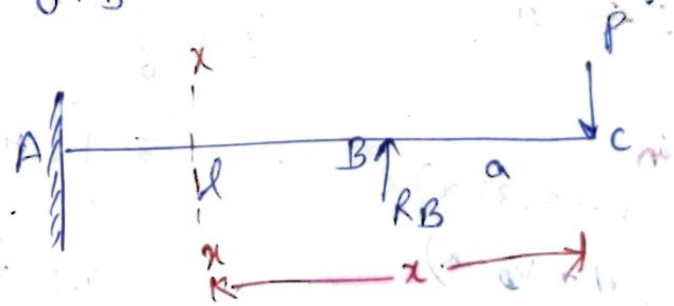
Ques




① Member CB

$$M_x = -P_x$$

$$\frac{\partial M_x}{\partial R_B} = 0$$




② Member BA

$$M_x = R_B \cdot x - P(a+x)$$

$$\frac{\partial M_x}{\partial R_B} = x$$

$$\int_0^a \cancel{\frac{(-P_x)\cdot 0 \, dx}{EI}}^{\,0} + \int_0^l \frac{(R_B \cdot x - P(a+x))}{EI} \cdot x \, dx$$

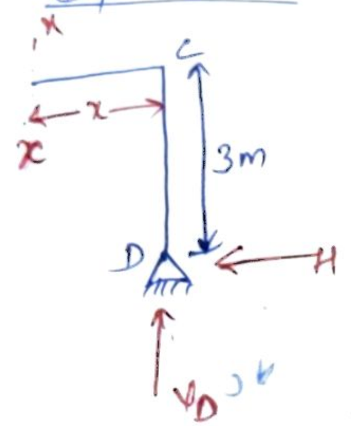
$$= 0$$

$$= \int_0^1 \frac{(6.67x - 3H)(-3)\,dx}{EI} = \frac{-1}{EI}\int_0^1 (20x - 9H)\,dx$$

$$= \frac{-1}{EI}\left[\left(20\cdot\frac{x^2}{2}\right)_0^1 - \left(9H\cdot x\right)_0^1\right]$$

$$= \frac{-1}{EI}(10 - 9H)$$

**Span EC**



$$\frac{\partial U_{EC}}{\partial H} = \frac{1}{EI}\int_0^2 M_x \frac{\partial M_x}{\partial H}\,dx$$

$$= \frac{1}{EI}\int_0^2 (-3H + 3.33x)(-3)\,dx$$

$$= \frac{1}{EI}\int_0^2 (9H - 10x)\,dx$$

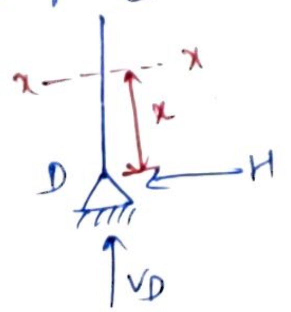
$$M_x = (-3H + 3.33x)$$
$$= (-3H + V_D \cdot x)$$

$$= \frac{1}{EI}\left[9H\left[x\right]_0^2 - 10\left[\frac{x^2}{2}\right]_0^2\right]$$

$$\frac{\partial M_x}{\partial H} = -3$$

$$= \frac{1}{EI}(18H - 20)$$

**Span CD**



$$\frac{\partial U_{CD}}{\partial H} = \frac{1}{EI}\int_0^3 Hx \cdot x\,dx.$$

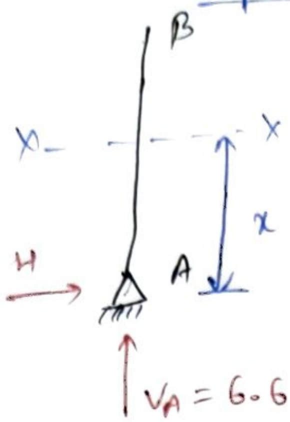
$$= \frac{H}{EI}\left[\frac{x^3}{3}\right]_0^3 = \frac{9H}{EI}$$

$$\boxed{\frac{\partial U}{\partial H} = 0}$$

$$\frac{1}{EI}\left[9H + 9H - 10 + 18H - 20 + 9H\right] = 0$$

$$\boxed{H = 0.667\ \text{kN}}$$

Span AB




$$\frac{\partial U_{AB}}{\partial H} = \frac{1}{EI}\int_0^l M_x \frac{\partial M_x}{\partial H} \cdot dx$$

$$M_x = -Hx$$

$$\frac{\partial M_x}{\partial H} = -x$$

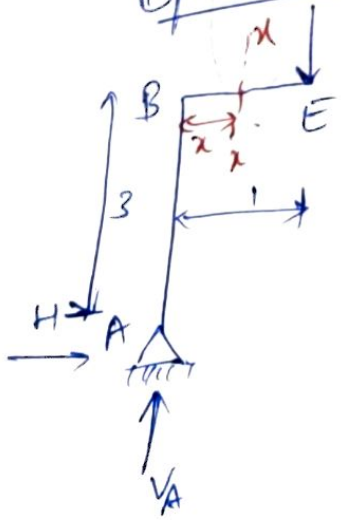
$$\frac{\partial U_{AB}}{\partial H} = \frac{1}{EI}\int_0^3 (-H \cdot x)\cdot(-x)\, dx$$

$$= \frac{1}{EI} \cdot \int_0^3 H x^2 \, dx$$

$$= \frac{H}{EI}\left[\frac{x^3}{3}\right]_0^3$$

$$= \frac{H}{EI} \cdot \frac{3 \times 9}{3} = \frac{9H}{EI}$$

Span BE

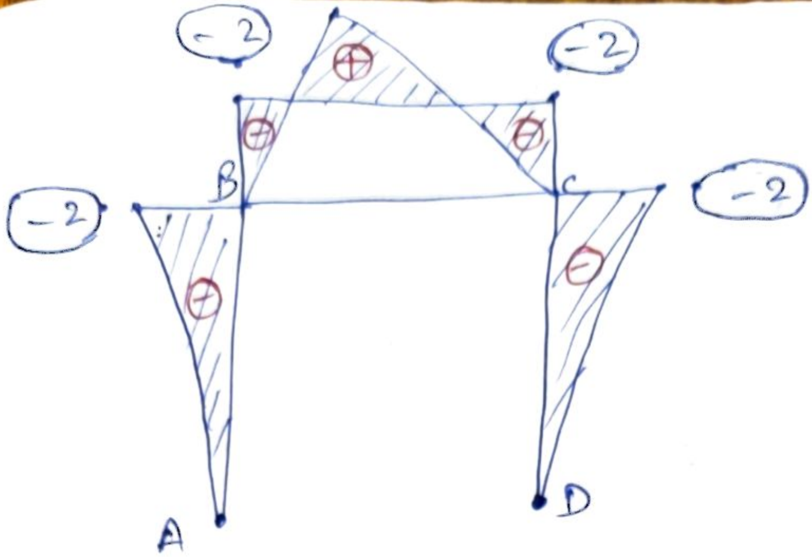


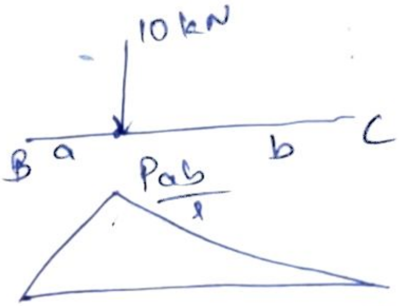

$$\frac{\partial U_{BE}}{\partial H} = \int_0^l \frac{M_x \cdot \frac{\partial M_x}{\partial H}}{EI} \cdot dx$$

$$M_x = V_A \cdot x - 3H$$

$$\frac{\partial M_x}{\partial H} = -3$$

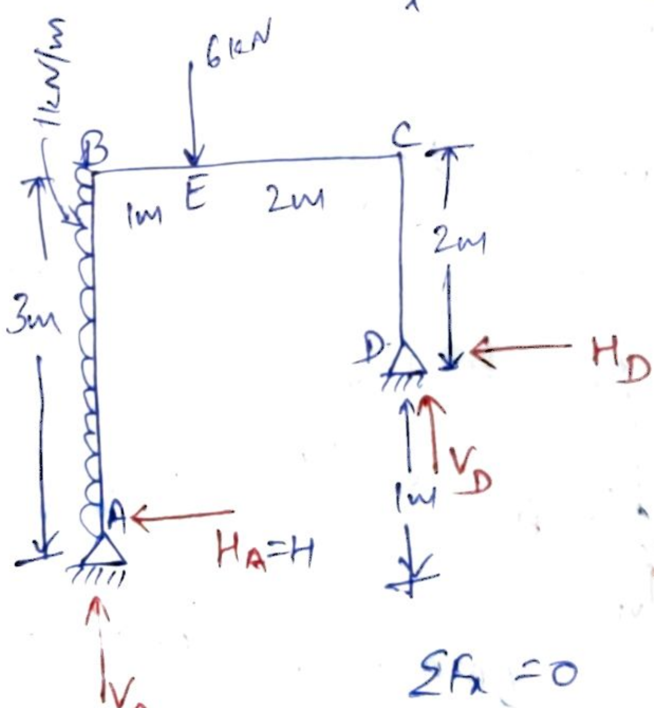
$$\frac{\partial U_{BE}}{\partial H} = \int_0^l \frac{(V_A \cdot x - 3H)(-3) \cdot dx}{EI}$$




$$\frac{Pab}{l} = \frac{10\times1\times2}{3} = 6.67$$

Ques




$\Sigma F_x = 0$

$H_A + H_D = 3$

$H + H_D = 3$

$$\boxed{H_D = 3 - H}$$

$\Sigma M_D = 0$

$V_A \times 3 + H \times 1 + 3 \times 0.5 - 6 \times 2 = 0$

$V_A = \frac{12 - 1.5 - H}{3} = \frac{10.5 - H}{3}$

$$\boxed{V_A = 3.5 - \frac{H}{3}}$$

$\Sigma F_y = 0$

$V_A + V_D = 6\ kN$

$D_s = r_e - E$
$= 4 - 3$
$= 1$

$3.5 - \frac{H}{3} + V_D = 6$

$$\boxed{V_D = 2.5 + \frac{H}{3}}$$

H be the redundant reaction.

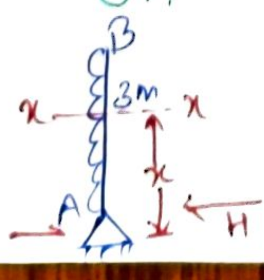
* Principal of Minimum Strain Energy Theorem

$$\boxed{\frac{\partial U}{\partial H} = 0}$$

$$\frac{\partial U_{AB}}{\partial H} + \frac{\partial U_{BE}}{\partial H} + \frac{\partial U_{EC}}{\partial H} + \frac{\partial U_{CD}}{\partial H} = 0$$

Span AB

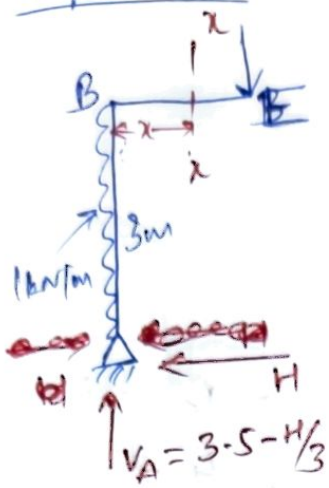
$$\frac{\partial U_{AB}}{\partial H} = \frac{1}{EI} \int_0^3 M_x \frac{\partial M_x}{\partial H} \cdot dx$$




$M_x = Hx - x^2/2$

$\frac{\partial M_x}{\partial H} = x$

## Span BE

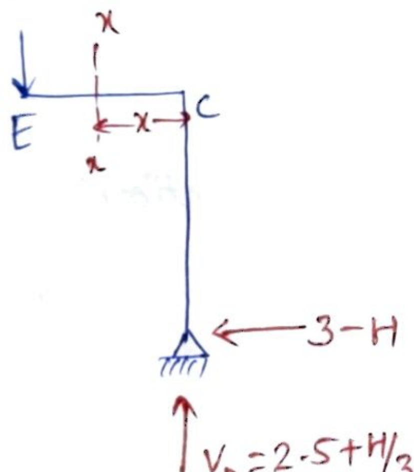



$$M_x = V_A \cdot x + 3H - \frac{1 \times (3)^2}{2}$$

$$M_x = V_A \cdot x + 3H - 4.5$$

$$\frac{\partial M_x}{\partial H} = -\frac{x}{3} + 3 = 3 - x/3$$

## Span EC

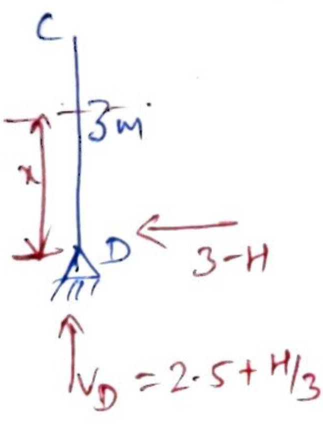



$$M_x = V_D \cdot x - (3-H)2$$

$$= (2.5 + H/3)x - 6 + 2H$$

$$\frac{\partial M_x}{\partial H} = \frac{x}{3} + 2$$

## Span CD




$$M_x = -(3-H)x$$

$$\frac{\partial M_x}{\partial H} = x$$

## Advantages of FB over SSB

1. Lesser magnitude of bending moments.
2. Lesser magnitude of maximum deflections.

## Disadvantages of FB compared to SSB

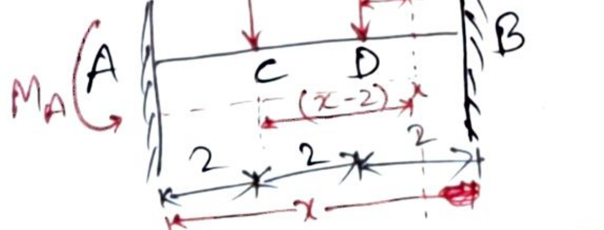
1. Due to fixidity, there is no space for expansion due to temperature rise.
2. Special care has to be undertaken, when aligning the supports horizontally.

**Example 1:** For the given fixed beam

Calculate : (i) fixing Moments

(ii) Draw SFD & BMD

(iii) Max Deflection.




$EI = 16 \times 10^4 \ kNm^2$

Section from A : $\Sigma M = 0$

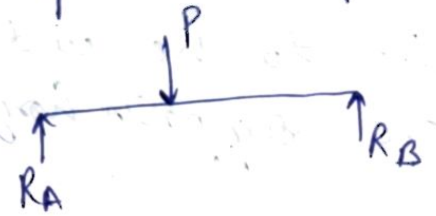
$$M_x - R_A \cdot x + M_A + 150(x-2) + 150(x-4) = 0.$$

$$M_x = R_A \cdot x - M_A - 150(x-2) - 150(x-4)$$

# Fixed Beam

* Important points about Fixed Beam

1. It is a beam whose both ends are fixed.
2. It is statically indeterminate


$\Sigma f_y = 0$

$\Sigma M_A = 0$

Fixed Beam :
A beam which is built in at its two supports is called a constrained beam (or) fixed beam.

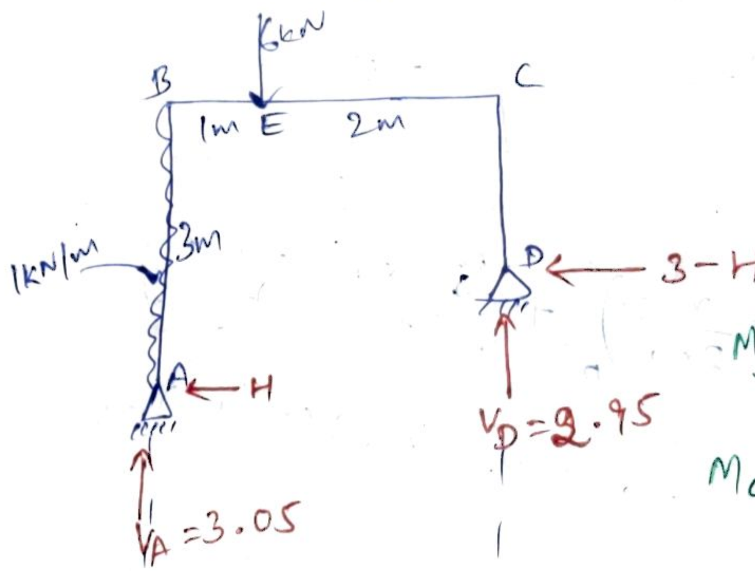
3. Also known as a built in / encastre beam as it is extended into the supports.
4. Due to fixidity the slope and deflection at both ends are zero.
5. Nature of fixing moments is hogging.
6. In a fixed beam, there are 4 unknowns ($R_A$, $R_B$, $M_A$ & $M_B$).

$$+\left(\frac{Hx^3}{3}-\frac{3x^3}{3}\right)_0^2 = 0$$

$$\boxed{H = 1.36\ kN}$$

$$V_A = 3.5 - \frac{H}{3} = 3.5 - \frac{1.36}{3} = 3.05$$

$$V_D = 2.5 + \frac{H}{3} = 2.5 + \frac{1.36}{3} = 2.95\ kN$$




$M_A = 0$

$M_D = 0$

$M_B = H \times 3 - \frac{1 \times (3)^2}{2}$

$M_C = -(3 - H) \times 2$

$(3)^2$

$5$

$x/3$

$+ 2H$

$2$

$$\frac{\partial U_{AB}}{\partial H} + \frac{\partial U_{BE}}{\partial H} + \frac{\partial U_{EC}}{\partial H} + \frac{\partial U_{CD}}{\partial H} = 0$$

$$\frac{1}{EI}\left[\int_0^3 (Hx - x^2/2)x\,dx + \int_0^1 ((3.5 - H/3)x + 3H - 4.5)(3 - x/3)\,dx\right.$$

$$+ \int_0^2 ((2.5 + H/3)x - 6 + 2H)(x/3 + 2)\,dx$$

$$\left. + \int_0^2 (H-3)x \cdot x\,dx\right] = 0$$

$$\frac{1}{EI}\left[\left(\frac{Hx^3}{3} - \frac{x^4}{2\times 4}\right)_0^3 + \cancel{\left(\frac{3.5x^2}{2} - \frac{Hx^2}{6}\right)}\right.$$

$$\int_0^1 (3.5x - \frac{Hx}{3} + 3H - 4.5)(3 - x/3)\,dx$$

$$+ \int_0^2 (2.5x + \frac{Hx}{3} - 6 + 2H)(x/3 + 2)\,dx$$

$$+ \int_0^2 Hx^2 - 3x^2\,dx = 0$$

$$\frac{1}{EI}\left[(9H - \frac{81}{6}) + \left(10.5\frac{x^2}{2} - \frac{\cancel{H}x^2}{2} + 9Hx\right.\right.$$

$$\left. - 13.5x - \frac{3.5x^3}{3\times 3} + \frac{Hx^3}{3\times 3} + \frac{\cancel{H}x^2}{2} + \frac{4.5x^2}{3\times 2}\right)_0^1$$

$$+ \left(\frac{5x^2}{2} + \frac{2}{3}\frac{Hx^2}{2} - 12x + 4Hx + \frac{2.5x^3}{3\times 3} + \frac{H}{9}\cdot\frac{x^3}{3} - \frac{2x^2}{2} + \frac{2H}{3}\cdot\frac{x^2}{2}\right)_0^2$$

$$EI\frac{d^2y}{dx^2} = R_A \cdot x - M_A - 150(x-2) - 150(x-4)$$

$$EI\frac{dy}{dx} = R_A \cdot \frac{x^2}{2} + M_A \cdot x + C_1 \Big| - \frac{150(x-2)^2}{2} - \frac{150(x-4)^2}{2}$$

$$EI\,y = R_A\frac{x^3}{3} - \frac{M_A x^2}{2} + C_1 x + C_2 \Big| - \frac{150(x-2)^3}{6} \Big| - \frac{150(x-4)^3}{3}$$

* **Boundary Condition :**

(1) $x = 0$, $\frac{dy}{dx} = 0 \Rightarrow C_1 = 0$

(2) $x = 0$, $y = 0$, $C_2 = 0$.

Slope Equation.

$$EI\frac{dy}{dx} = \frac{R_A x^2}{2} - M_A \cdot x \Big| - \frac{150(x-2)^2}{2} \Big| - \frac{150(x-4)^2}{2}$$

**Deflection Eq$^n$.**

$$EI \cdot y = R_A\frac{x^3}{3} - \frac{M_A \cdot x^2}{2} \Big| - \frac{150(x-2)^3}{6} \Big| - \frac{150(x-4)^3}{6}$$

Boundary Conditions

$x=0, \frac{dy}{dx}=0, C_1=0$

$$EI \cdot y = R_A \cdot \frac{x^3}{6} - M_A \cdot \frac{x^2}{2} - \frac{20x^4}{12} - \frac{40(x-6)^3}{3}$$

$$+\frac{20(x-4)^4}{12} + C_1 x + C_2$$

at $x=0, y=0, C_2=0$

At $x=8, \frac{dy}{dx}=0$

$32R_A - 8M_A = 3146.66$ — (1)

At $x=8, y=0$

$85.33R_A - 32M_A = 6506.66$ — (2)

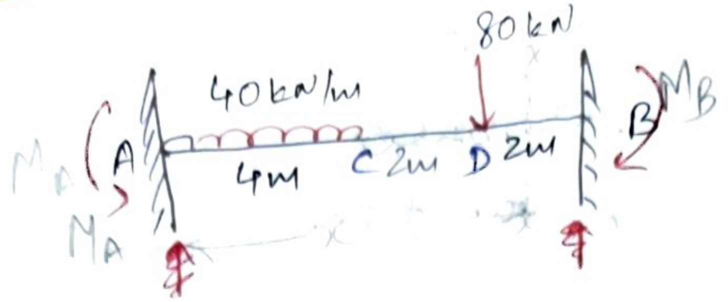
$R_A = 142.5$ kN, $M_A = 176.6$ kN

$\Sigma F_y = 0$

$\cancel{R_A}$ $\boxed{R_B = 97.5 \text{ kN}}$

$142.5 + R_B - 80 - 160 = 0$

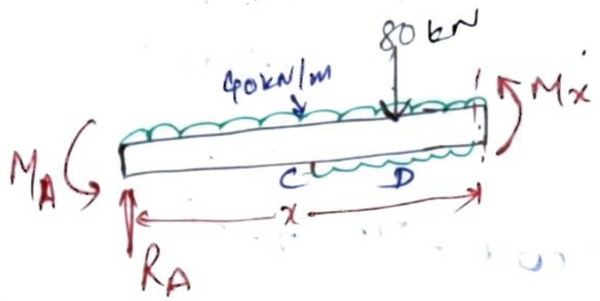
$\Sigma M_A = 0$




UDL must finish on the section

Yes (no issue)

No, (Extend the UDL from top & bottom)




**Sol:** $\Sigma M = 0$

$\cancel{M_x = R_A \cdot x + M}$

$$R_A \cdot x - M_A - \frac{40x^2}{2} - 80(x-6) + \frac{40(x-4)^2}{2} = M_x$$

$$EI\frac{d^2y}{dx^2} = R_A \cdot x - M_A - 20x^2 - 80(x-6) + 20(x-4)^2$$

$$EI\frac{dy}{dx} = \frac{R_A \cdot x^2}{2} - M_A \cdot x - \frac{20x^3}{3} - \frac{80(x-6)^2}{2} + \frac{20(x-4)^3}{3} + C_1$$

$$\boxed{\begin{array}{ll} 0 \le x \le 2 & - 1^{st}\ 2\ \text{terms} \\ 2 \le x < 4 & - 1^{st}\ 3\ \text{terms} \\ 4 \le x < 6 & - \text{All the terms} \end{array}}$$

Maximum Deflection.

↳ Between C & D $\underbrace{2 \le x < 4}_{\text{First 3 terms}}$

$$\frac{dy}{dx} = 0$$

$$0 = 75x^2 - 200x - 75(x-2)^2$$

$$75x^2 - 200x - 75x^2 - 300 + 300x = 0$$

$$100x = 300$$

$$x = 3\ m.$$

$$y_{max\,(x=3)} =$$

* at End B ⇒ at $x = 6$, $\frac{dy}{dx} = 0$.

$18 R_A - 6 M_A - 1200 - 300 = 0$

$\therefore 18 R_A - 6 M_A = 1500$ —— ①

at $x = 6$, $y = 0$.

$36 R_A - 18 M_A - 1600 - 200 = 0$ —— ②

Solve the equations ① & ②

$R_A = 150$ kN

$M_A = 200$ kNm

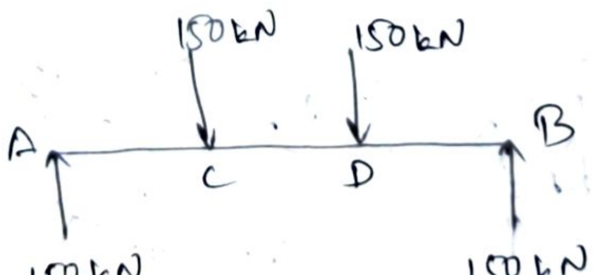
$\Sigma F_y = 0$; $150 - 300 + R_B = 0$

$\therefore R_B = 150$ kN

$\Sigma M_A = 0$ ; $200 - (150 \times 2) - (150 \times 4) - M_B + 6 R_B = 0$

$200 - 300 - 600 - M_B + 900 = 0$

$M_B = 200$ kNm.




Moment at

$M_C = 150 \times 2 = 300$ kNm

$M_D = 150 \times 2 = 300$ kNm